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Abstract

We consider the problem of efficient estimation for the drift of fractional Brownian motion $B^H := \left(B_t^H\right)_{t\in[0,T]}$ with hurst parameter H less than $\frac{1}{2}$. We also construct superefficient James-Stein type estimators which dominate, under the usual quadratic risk, the natural maximum likelihood estimator.

Key words : Fractional Brownian Motion, Stein estimate, MLE

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1 Introduction

Fix $H \in (0,1)$ and $T > 0$. Let $B^H = \left\{(B_t^{H,1}, ..., B_t^{H,d}); t \in [0,T]\right\}$ be a d-dimensional fractional Brownian motion (fBm) defined on the probability space $(\Omega, \mathcal{F}, P)$. That is, B^H is a zero mean Gaussian vector whose components are independent one-dimensional fractional Brownian motions with Hurst parameter $H \in (0,1)$, i.e., for every $i = 1, ..., d$ $B^{H,i}$ is a Gaussian process and covariance function given by

$$E(B_s^{H,i} B_t^{H,i}) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T].$$

For each $i = 1, \ldots, d$, $\left(\mathcal{F}_t^i\right)_{t\in[0,T]}$ denotes the filtration generated by $\left(B_t^{H,i}\right)_{t\in[0,T]}$. The fBm was first introduced by [5] and studied by [6]. Notice that if $H = \frac{1}{2}$, the process $B^{\frac{1}{2}}$ is a standard Brownian motion. However, for $H \neq \frac{1}{2}$, the fBm is neither a Markov process, nor a semi-martingale.

Let M be a subspace of the Cameron-Martin space defined by

$$M = \Big\{ \varphi : [0,T] \to \mathbb{R}^d; \varphi_t^i = \int_0^t \dot{\varphi}_s^i ds \text{ with } \dot{\varphi}^i \in L^2([0,T]) \\ \text{and } \varphi^i \in I_{0^+}^{H+\frac{1}{2}} \left(L^2([0,T]) \right), i = 1, ..., d \Big\}.$$

Let $\theta = \left\{ (\theta_t^1, \dots, \theta_t^d); t \in [0,T] \right\}$ be a function belonging to M. Then, Applying Girsanov theorem (see Theorem 2 in [9]), there exist a probability measure absolutely continuous with respect to P under which the process $\widetilde{B}^H$ defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0,T] \tag{1.1}$$

is a fBm with Hurst parameter H and mean zero. In this case, we say that, under the probability P_θ, the process B^H is a fBm with drift θ.

We consider in this paper the problem of estimating the drift θ of B^H under the probability P_θ, with hurst parameter $H < 1/2$. We wish to estimate θ under the usual quadratic risk, that is defined for any estimator δ of θ by

$$\mathcal{R}(\theta, \delta) = E_\theta \left[\int_0^T ||\delta_t - \theta_t||^2 dt \right]$$

where E_θ is the expectation with respect to a probability P_θ.

Let $X = (X^1, \dots, X^d)$ be a normal vector with mean $\theta = (\theta^1, \dots, \theta^d) \in \mathbb{R}^d$ and identity covariance matrix $\sigma^2 I_d$. The usual maximum likelihood estimator of θ is X itself. Moreover, it is efficient in the sense that the Cramer-Rao bound over all unbiased estimators is attained by X. That is

$$\sigma^2 d = E\left[\|X - \theta\|_d^2 \right] = \inf_{\xi \in \mathcal{S}} E\left[\|\xi - \theta\|_d^2 \right],$$

where $\mathcal{S}$ is the class of unbiased estimators of θ and $\|.\|_d$ denotes the Euclidean norm on $\mathbb{R}^d$.

[12] constructed biased superefficient estimators of θ of the form

$$\delta_{a,b}(X) = \left(1 - \frac{b}{a + ||X||^2} \right) X$$

for a sufficiently small and b sufficiently large when $d \geq 3$. [4] sharpened later this result and presented an explicit class of biased superefficient estimators of the form

$$\left(1 - \frac{a}{||X||_d^2}\right) X, \text{ for } 0 < a < 2(d-2).$$

Recently, an infinite-dimensional extension of this result has been given by [10]. The authors constructed unbiased estimators of the drift $(\theta_t)_{t\in[0,T]}$ of a continuous Gaussian martingale $(X_t)_{t\in[0,T]}$ with quadratic variation $\sigma_t^2 dt$, where $\sigma \in L^2([0,T], dt)$ is an a.e. non-vanishing function. More precisely, they proved that $\hat{\theta} = (X_t)_{t\in[0,T]}$ is an efficient estimator of $(\theta_t)_{t\in[0,T]}$. On the other hand, using Malliavin calculus, they constructed superefficient estimators for the drift of a Gaussian processe of the form:

$$X_t := \int_0^t K(t,s) dW_s, \qquad t \in [0,T],$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion and $K(.,.)$ is a deterministic kernel. These estimators are biased and of the form $X_t + D_t \log F$, where F is a positive superharmonic random variable and D is the Malliavin derivative.

In Section 3, we prove, using technic based on the fractional calculus and Girsanov theorem, that $\widehat{\theta} = B^H$ is an efficient estimator of θ under the probability P_θ with risk

$$\mathcal{R}(\theta, B^H) = E_\theta \left[\int_0^T \|B_t^H - \theta_t\|^2 dt \right] = \frac{T^{2H+1}}{2H+1} d.$$

Moreover, we will establish that $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ.

In Section 4, we construct a class of biased estimators of James-Stein type of the form

$$\delta(B^H)_t = \left(1 - at^{2H} \left(\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right)\right) B_t^H, \qquad t \in [0,T].$$

We give sufficient conditions on the function r and on the constant a in order that $\delta(B^H)$ dominates B^H under the usual quadratic risk i.e.

$$\mathcal{R}\left(\theta, \delta\left(B^H\right)\right) < \mathcal{R}\left(\theta, B^H\right) \qquad \text{for all } \theta \in M. \tag{1.2}$$

2 Preliminaries

This section contains the elements from fractional calculus that we will need in the paper.

The fractional Brownian motion B^H has the following stochastic integral representation (see for instance, [1], [8])

$$B_t^{H,i} = \int_0^t K_H(t,s)dW_s^i, \quad i = 1, ..., d; \quad t \in [0,T] \tag{2.3}$$

where $W = (W^1, ..., W^d)$ denotes the d-dimensional Brownian motion and the kernel $K_H(t,s)$ is equal to

$$c_H(t-s)^{H-\frac{1}{2}} + c_H(\frac{1}{2} - H) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - (\frac{s}{u})^{\frac{1}{2}-H}\right) du \quad \text{if } H \le \frac{1}{2}$$
$$c_H(H - \frac{1}{2}) \int_s^t (u-s)^{H-\frac{3}{2}} \left(\frac{s}{u}\right)^{H-\frac{1}{2}} du \quad \text{if } H > \frac{1}{2},$$

if $s < t$ and $K_H(t,s) = 0$ if $s \ge t$. Here c_H is the normalizing constant

$$c_H = \sqrt{\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}}$$

where Γ is the Euler function.
We recall some basic definitions and results on classical fractional calculus which we will need. General information about fractional calculus can be found in [11].
The left fractional Riemann-Liouville integral of $f \in L^1((a,b))$ of order $\alpha > 0$ on (a,b) is given at almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)dy.$$

If $f \in I_{a+}^{\alpha}(L^p(a,b))$ with $0 < \alpha < 1$ and $p > 1$ then the left-sided Riemann-Liouville derivative of f of order α defined by

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right)$$

for almost all $x \in (a,b)$.
For $H \in (0,1)$, the integral transform

$$(K_H f)(t) = \int_0^t K_H(t,s) f(s) ds$$

is a isomorphism from $L^2([0,1])$ onto $I_{0+}^{H+\frac{1}{2}}\left(L^2([0,1])\right)$ and its inverse operator K_H^{-1} is given by

$$K_H^{-1}f = t^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}f' \quad \text{for } H > 1/2, \tag{2.4}$$

$$K_H^{-1}f = t^{\frac{1}{2}-H}D_{0+}^{\frac{1}{2}-H}t^{H-\frac{1}{2}}D_{0+}^{2H}f \quad \text{for } H < 1/2. \tag{2.5}$$

Moreover, for $H < \frac{1}{2}$, if f is an absolutely continuous function then $K_H^{-1}f$ can be represented of the form (see [9])

$$K_H^{-1}f = t^{H-\frac{1}{2}}I_{0+}^{\frac{1}{2}-H}t^{\frac{1}{2}-H}f'. \tag{2.6}$$

3 The maximum likelihood estimator and Cramer-Rao type bound

We consider a function $\theta = \left(\theta^1, \dots, \theta^d\right)$ belonging to M. An estimator $\xi = (\xi^1, \dots, \xi^d)$ of $\theta = (\theta^1, \dots, \theta^d)$ is called unbiased if, for every $t \in [0,T]$

$$E_\theta(\xi_t^i) = \theta_t^i, \qquad i = 1, \dots, d$$

and it is called adapted if, for each $i = 1, \dots, d$, ξ^i is adapted to $\left(\mathcal{F}_t^i\right)_{t\in[0,T]}$.
Since for any $i = 1, ..., d$, the function θ^i is deterministic and

$$\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds < \infty,$$

then Girsanov theorem yields that there exists a probability measure P_θ absolutely continuous with respect to P under which the process $\widetilde{B}^H := \left(\widetilde{B}_t^H; t \in [0,T]\right)$ defined by

$$\widetilde{B}_t^H = B_t^H - \theta_t, \qquad t \in [0,T] \tag{3.7}$$

is a d-dimensional fBm with Hurst parameter H and mean zero. Moreover the Girsanov density of P_θ with respect to P is given by:

$$\frac{dP_\theta}{dP} = \exp\left[\sum_{i=1}^d \left(\int_0^T K_H^{-1}(\theta^i)(s)dW_s^i - \frac{1}{2}\int_0^T (K_H^{-1}(\theta^i)(s))^2 ds\right)\right]$$

and

$$\widetilde{B}_t^H = \int_0^t K_H(t,s)d\widetilde{W}_s$$

where $\widetilde{W}$ is a d-dimensional Brownian motion under the probability P_θ and

$$\widetilde{W}_t^i = W_t^i - \int_0^t K_H^{-1}(\theta^i)(s)ds, \quad i = 1, ..., d; \quad t \in [0,T].$$

The equation (3.7) implies that B^H is an unbiased and adapted estimator of θ under probability P_θ. In addition, we obtain the Cramer-Rao type bound:

$$R(H,\hat{\theta}) := \mathcal{R}(\theta, B^H) = \int_0^T E_\theta \|\widetilde{B}_t^H\|^2 dt = d\int_0^T t^{2H} dt = \frac{T^{2H+1}}{2H+1} d.$$

The first main result of this section is given by the following proposition which asserts that $\widehat{\theta} = B^H$ is an efficient estimator of θ.

Theorem 1 *Assume that $H < \frac{1}{2}$. If ξ is an unbiased and adapted estimator of θ, then*

$$E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt \geq R(H,\hat{\theta}). \tag{3.8}$$

Proof: Since ξ is unbiased, then for every $\varphi \in M$ we have

$$E_\varphi(\xi_t^j) = E_\varphi(\varphi_t^j), \qquad j = 1, \ldots, d.$$

Let $\varphi = \theta + \varepsilon\psi$ with $\psi \in M$ and $\varepsilon \in \mathbb{R}$. Then for every $t \in [0,T]$ and $j \in \{1, \ldots, d\}$, we have

$$\begin{aligned} E_{\theta+\varepsilon\psi}(\xi_t^j) &= E_{\theta+\varepsilon\psi}(\theta_t^j + \varepsilon\psi_t^j) \\ &= E_{\theta+\varepsilon\psi}(\theta_t^j) + \varepsilon\psi_t^j. \end{aligned}$$

This implies that for every $j = 1, \ldots, d$

$$
\begin{aligned}
\psi_t^j &= \frac{d}{d\varepsilon}_{/\varepsilon=0} E_{\theta+\varepsilon\psi}(\xi_t^j - \theta_t^j) \\
&= E\left(\frac{d}{d\varepsilon}_{/\varepsilon=0} \exp\left[\sum_{i=1}^d \left(\int_0^t K_H^{-1}(\theta^i + \varepsilon\psi^i)(s) dW_s^i \right.\right.\right. \\
&\qquad \left.\left.\left. -\frac{1}{2}\int_0^t (K_H^{-1}(\theta^i + \varepsilon\psi^i)(s))^2) ds \right)\right] (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left(\sum_{i=1}^d \left[\int_0^t K_H^{-1}(\psi^i)(s) dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s) K_H^{-1}(\theta^i)(s) ds \right] \right. \\
&\qquad \left. \times (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left(\sum_{i=1}^d \left[\int_0^t K_H^{-1}(\psi^i)(s) d\widetilde{W}_s^i \right] (\xi_t^j - \theta_t^j) \right) \\
&= E_\theta \left(\left[\int_0^t K_H^{-1}(\psi^j)(s) d\widetilde{W}_s^j \right] (\xi_t^j - \theta_t^j) \right).
\end{aligned}
$$

Applying Cauchy-Schwarz inequality in $L^2(\Omega, dP_\theta)$, we obtain that for every $t \in [0, T]$

$$
\begin{aligned}
\|\psi_t\|^2 = \sum_{j=1}^d (\psi_t^j)^2 \quad &\leq \quad \sum_{j=1}^d E_\theta \left((\xi_t^j - \theta_t^j)^2 \right) E_\theta \left(\left[\int_0^t K_H^{-1}(\psi^j)(s) d\widetilde{W}_s^j \right]^2 \right) \\
&= \sum_{j=1}^d E_\theta \left[\left((\xi_t^j - \theta_t^j)^2 \right) \int_0^t (K_H^{-1}(\psi^j)(s))^2 ds \right].
\end{aligned}
$$

We take for each $j = 1, \ldots, d$, $\psi_t^j = t^{2H}$. Since $t \longrightarrow t^{2H}$ is absolutely continuous function, then by (2.6), a simple calculation shows that

$$
\begin{aligned}
K_H^{-1}(t^{2H}) &= 2Ht^{H-\frac{1}{2}} I_{0+}^{\frac{1}{2}-H} t^{H-\frac{1}{2}} \\
&= \frac{2H\beta(\frac{1}{2} - H, H + \frac{1}{2})}{\Gamma(\frac{1}{2} - H)} t^{H-1/2} \\
&= 2H(\Gamma(\frac{1}{2} + H)) t^{H-1/2}.
\end{aligned}
$$

It is known that

$$
0 \leq \Gamma(z) \leq 1 \quad \text{for every } z \in [1, 2]. \tag{3.9}
$$

Combining the facts that $z\Gamma(z) = \Gamma(z+1)$, $z > 0$, $2H \leq (H + \frac{1}{2})^2$ and (3.9), we obtain

$$dt^{2H} = \|\psi_t\|^2 \quad \leq \quad (\Gamma(\frac{3}{2} + H))^2 E_\theta\left(\|\xi_t - \theta_t\|^2\right) \leq E_\theta\left(\|\xi_t - \theta_t\|^2\right).$$

Hence, by an integration with respect to dt, we get

$$R(H, \hat{\theta}) = \frac{T^{2H+1}}{2H+1} \leq E_\theta \int_0^T \|\xi_t - \theta_t\|^2 dt.$$

Therefore (3.8) is satisfied.

Corollary 1 *The process $\hat{\theta} = B^H$ is a maximum likelihood estimator of θ.*

Proof: We have for every $\psi \in M$

$$\frac{d}{d\varepsilon}_{/\varepsilon=0} \exp\left[\sum_{i=1}^d \int_0^t K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s)dW_s^i - \frac{1}{2}\int_0^t (K_H^{-1}(\hat{\theta}^i + \varepsilon\psi^i)(s))^2)ds\right] = 0.$$

Hence

$$\sum_{i=1}^d \left(\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s)ds\right) = 0.$$

Which implies that for every $i = 1, ..., d$

$$E\left(\int_0^t K_H^{-1}(\psi^i)(s)dW_s^i - \int_0^t K_H^{-1}(\psi^i)(s)K_H^{-1}(\hat{\theta}^i)(s)ds\right)^2 = 0.$$

Then, for each $i = 1, ..., d$

$$W_t^i = \int_0^t K_H^{-1}(\hat{\theta}^i)(s)ds, \qquad t \in [0, T].$$

Therefore by (2.3), we obtain that $B^H = \hat{\theta}$.

4 Superefficient James-Stein type estimators

The aim of this section is to construct superefficient estimators of θ which dominate, under the usual quadratic risk, the natural MLE estimator B^H. The class of estimators considered here are of the form

$$\delta(B^H)_t = B_t^H + g(B_t^H, t), \qquad t \in [0, T] \tag{4.10}$$

where $g : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ is a function. The problem turns to find sufficient conditions on g such that $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$ and the risk difference is negative, i.e.

$$\Delta\mathcal{R}(\theta) = \mathcal{R}\left(\theta, \delta(B^H)\right) - \mathcal{R}\left(\theta, B^H\right) < 0.$$

In the sequel we assume that the function g satisfies the following assumption:

$$(A)\begin{cases} E_\theta\left[\int_0^T ||g(B_t^H, t)||_d^2 \, dt\right] < \infty, \\ \text{the partial derivatives } \partial_i g^i := \frac{\partial g^i}{\partial x^i}, \ i = 1, \ldots, n \text{ of } g \text{ exist.} \end{cases}$$

Then $\mathcal{R}\left(\theta, \delta(B^H)\right) < \infty$. Moreover

$$\begin{aligned} \Delta\mathcal{R}(\theta) &= E_\theta\left[\int_0^T ||B_t^H + g(B_t^H, t) - \theta_t||_d^2 - ||B_t^H - \theta_t||_d^2 dt\right] \\ &= E_\theta\left[\int_0^T ||g(B_t^H, t)||_d^2 + 2\sum_{i=1}^d \left(g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i)\right) dt\right]. \end{aligned}$$

In addition,

$$\begin{aligned} & E_\theta \int_0^T \sum_{i=1}^d \left(g^i(B_t^H, t)(B_t^{H,i} - \theta_t^i)\right) dt \\ = & \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} g^i(x^1, \ldots, x^d, t)(x^i - \theta_t^i)\right. \\ & \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)2}{2t^{2H}}} dx^1 \ldots dx^d\right) dt \\ = & \sum_{i=1}^d \int_0^T (2\pi t^{2H})^{-\frac{d}{2}} \left(\int_{\mathbb{R}^d} t^{2H} \partial_i g^i(x^1, \ldots, x^d, t)\right. \\ & \left. \times e^{-\frac{\sum_{j=1}^d (x^j - \theta_t^j)2}{2t^{2H}}} dx^1 \ldots dx^d\right) dt \\ = & \sum_{i=1}^d \int_0^T \left(t^{2H} E_\theta \partial_i g^i(B_t^H, t)\right) dt = E_\theta\left[\sum_{i=1}^d \int_0^T t^{2H} \partial_i g^i(B_t^H, t) dt\right]. \end{aligned}$$

Consequently, the risk difference equals

$$\Delta\mathcal{R}(\theta) = E_\theta\left[\int_0^T \left(||g(B_t^H, t)||^2 + 2t^{2H} \sum_{i=1}^d \partial_i g^i(B_t^H, t)\right) dt\right]. \tag{4.11}$$

We can now state the following theorem.

Theorem 2 *Let $g : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ be a function satisfying (A). A sufficient conditions for the estimator $\left(B_t^H + g(B_t^H, t)\right)_{t\in[0,T]}$ to dominate B^H under the usual quadratic risk is*

$$E_\theta \left[\int_0^T \left(||g(B_t^H, t)||^2 + 2t^{2H} \sum_{i=1}^{d} \partial_i g^i(B_t^H, t) \right) dt \right] < 0.$$

As an application, take g of the form

$$g(x,t) = at^{2H} \frac{r\left(\|x\|^2\right)}{\|x\|^2} x, \tag{4.12}$$

where a is a constant strictly positive and $r : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded derivable function. The next lemma give a sufficient condition for g in (4.12) to satisfies the assumption (A).

Lemma 1 *If $d \geq 3$ and $H < \frac{1}{2}$ then*

$$E\left[\int_0^T \frac{1}{\|B_t^H\|^2} dt\right] < \infty. \tag{4.13}$$

Proof: Firstly the integral given by (4.13) is well defined, because

$$(dt \times P)\left((t,w); B_t^H(w) = 0\right) = 0$$

where $(dt \times P)$ is the product measure.
Using the change of variable and $d \geq 3$ we see that

$$E\int_0^T \frac{1}{\|B_t^H\|^2} dt = \int_0^T \frac{dt}{t^{2H}} \int_{\mathbb{R}^d} \frac{e^{-\frac{\|y\|^2}{2}}}{\sqrt{2\pi}\|y\|^2} dy \leq C \int_0^T \frac{1}{t^{2H}} dt,$$

where C is a constant depending only on d. Furthermore, since $H < \frac{1}{2}$ then (4.13) holds.

Theorem 3 *Assume that $d \geq 3$. If the function r, the constant a and the parameter H satisfy:*

i) $0 \leq r(.) \leq 1$

ii) $r(\cdot)$ *is differentiable and increasing* *iii)* $0 < a \leq 2(d-2)$ *and* $H < 1/2$,

then the estimator

$$\delta(B^H) = B_t^H - at^{2H}\frac{r\left(\|B_t^H\|^2\right)}{\|B_t^H\|^2}B_t^H, \qquad t \in [0,T].$$

dominates B^H.

Proof: It suffices to prove that $\Delta\mathcal{R}(\theta) < 0$. From (4.11) and the hypothesis $i)$ and $ii)$ we can write

$$\begin{aligned}
\Delta\mathcal{R}(\theta) &= aE_\theta\left[\int_0^T t^{4H}\left(\frac{ar^2(\|B_t^H\|^2)}{\|B_t^H\|^2} - 2(d-2)\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right.\right. \\
&\qquad\qquad\left.\left. - 4r'(\|B_t^H\|^2)\right)dt\right] \\
&\leq a\left[a-2(d-2)\right]E_\theta\left[a\int_0^T t^{4H}\frac{r(\|B_t^H\|^2)}{\|B_t^H\|^2}\right].
\end{aligned}$$

Combining this fact with the assumption $iii)$ yields that the risk difference is negative. Which proves the desired result.
For $r = 1$, we obtain a James-Stein type estimator:

Corollary 2 *Let* $d \geq 3$, $0 < H < \frac{1}{2}$ *and* $0 < a \leq 2(d-2)$. *Then the estimator*

$$\left(1 - \frac{at^{2H}}{\|B_t^H\|^2}\right)B_t^H, \qquad t \in [0,T]$$

dominates B^H.

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